The problem of the $U(1)$ axial symmetry and the chiral transition in QCD

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We discuss the role of the $U(1)$ axial symmetry for the phase structure of QCD at finite temperature. In particular, supported by recent lattice results, we analyse a scenario in which a $U(1)$–breaking condensate survives across the chiral transition. This scenario can be consistently reproduced using an effective Lagrangian model. The effects of the $U(1)$ chiral condensate on the slope of the topological susceptibility in the full theory with quarks are studied. Further information on the new $U(1)$ chiral order parameter is derived from the study (at zero temperature) of the radiative decays of the "light" pseudoscalar mesons in two photons.

1. Introduction

It is well known that at zero temperature the $SU(L)\otimes SU(L)$ chiral symmetry, in a QCD with L massless quarks, is broken spontaneously by the non–zero value of the so–called *chiral condensate*, $\langle\bar{q}q\rangle \equiv \sum_{i=1}^{L}\langle\bar{q}_i q_i\rangle$, and the L^2-1 $J^P=0^-$ mesons are just the Goldstone bosons associated with this breaking. At high temperatures the thermal energy breaks up the $q\bar{q}$ condensate, leading to the restoration of chiral symmetry above a certain critical temperature T_{ch}, defined as the temperature at which the condensate $\langle\bar{q}q\rangle$ goes to zero. Instead, the role of the $U(1)$ axial symmetry [1,2] for the finite temperature phase structure of QCD has been so far not well studied and it is still an open question of hadronic physics.

In the "Witten–Veneziano mechanism" [3,4] for the resolution of the $U(1)$ problem, a fundamental role is played by the so–called "topological susceptibility" in a QCD without quarks, i.e., in a pure Yang–Mills (YM) theory, in the large–N_c limit (N_c being the number of colours):

$$A = \lim_{k\to 0}\lim_{N_c\to\infty}\left\{-i\int d^4x e^{ikx}\langle TQ(x)Q(0)\rangle\right\}, \quad (1)$$

where $Q(x)=\frac{g^2}{64\pi^2}\varepsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma}$ is the so–called "topological charge density". This quantity enters into the expression for the mass of the η'. Therefore, in order to study the role of the $U(1)$ axial symmetry for the full theory at non–zero temperatures, one should consider the YM topological susceptibility $A(T)$ at a given temperature T, formally defined as in Eq. (1), where now $\langle\ldots\rangle$ stands for the expectation value in the full theory at the temperature T [5].

The problem of studying the behaviour of $A(T)$ as a function of the temperature T was first addressed, in lattice QCD, in Refs. [6–8]. Recent lattice results [9] (obtained for the $SU(3)$ pure–gauge theory) show that the YM topological susceptibility $A(T)$ is approximately constant up to the critical temperature T_{ch}, it has a sharp decrease above the transition, but it remains different from zero up to $\sim 1.2\ T_{ch}$. We recall that, in the Witten–Veneziano mechanism [3,4], a (no matter how small!) value different from zero for A is related to the breaking of the $U(1)$ axial symmetry, since it implies the existence of a *would–be* Goldstone particle with the same quantum numbers of the η'.

Another way to address the same question is to look at the behaviour at non–zero temperatures of the susceptibilities related to the propagators for the following meson channels [10] (we consider for simplicity the case of $L=2$ light flavours): the isoscalar ($I=0$) scalar channel $O_\sigma=\bar{q}q$; the isovector ($I=1$) scalar channel $\vec{O}_\delta=\bar{q}\frac{\vec{\tau}}{2}q$; the isovector ($I=1$) pseudoscalar channel $\vec{O}_\pi=i\bar{q}\gamma_5\frac{\vec{\tau}}{2}q$; the isoscalar ($I=0$) pseudoscalar channel $O_{\eta'}=i\bar{q}\gamma_5 q$. Under $SU(2)$ chi-

ral transformations, σ is mixed with π (and δ is mixed with η'). On the contrary, under $U(1)$ chiral transformations, π is mixed with δ (and σ is mixed with η'). In practice, one can construct, for each meson channel f, the corresponding chiral susceptibility

$$\chi_f = \int d^4x \ \langle T O_f(x) O_f^\dagger(0) \rangle, \qquad (2)$$

and then define two order parameters: $\chi_{SU(2)\otimes SU(2)} \equiv \chi_\sigma - \chi_\pi$, and $\chi_{U(1)} \equiv \chi_\delta - \chi_\pi$. If an order parameter is non–zero in the chiral limit, then the corresponding symmetry is broken. Present lattice data for these quantities seem to indicate that the $U(1)$ order parameter survives across T_{ch}, up to $\sim 1.2\ T_{ch}$, where the δ–π splitting is small but still different from zero [11–13]. In terms of the left–handed and right–handed quark fields ($q_{L,R} \equiv \frac{1}{2}(1 \pm \gamma_5)q$, with $\gamma_5 \equiv -i\gamma^0\gamma^1\gamma^2\gamma^3$) one has the following expression for the difference between the correlators for the δ^+ and π^+ channels:

$$\langle O_{\delta^+}(x) O_{\delta^+}^\dagger(0) \rangle - \langle O_{\pi^+}(x) O_{\pi^+}^\dagger(0) \rangle =$$
$$2\left[\langle \bar{u}_R d_L(x) \bar{d}_R u_L(0) \rangle + \langle \bar{u}_L d_R(x) \bar{d}_L u_R(0) \rangle \right]. \quad (3)$$

(The integral of this quantity is just equal to the $U(1)$ chiral susceptibility $\chi_{U(1)} = \chi_\delta - \chi_\pi$.) What happens below and above T_{ch}? Below T_{ch}, in the chiral limit $\sup(m_i) \to 0$, the left–handed and right–handed components of a given light quark flavour (*up* or *down*, in our case with $L = 2$) can be connected through the $q\bar{q}$ chiral condensate, giving rise to a non–zero contribution to the quantity (3) (i.e., to the quantity $\chi_{U(1)}$). But above T_{ch} the $q\bar{q}$ chiral condensate is zero: so, how can the quantity (3) (i.e., the quantity $\chi_{U(1)}$) be different from zero also above T_{ch}, as indicated by present lattice data? The only possibility in order to solve this puzzle seems to be that of requiring the existence of a genuine four–fermion local condensate, which is an order parameter for the $U(1)$ axial symmetry and which remains different from zero also above T_{ch}. This new condensate will be discussed in Section 2 and then we shall analyse some interesting phenomenological consequences deriving from this hypothesis [14].

2. The $U(1)$ chiral order parameter

Let us define the following temperatures:

T_χ: the temperature at which the pure–gauge topological susceptibility A drops to zero. Present lattice results indicate that $T_\chi \geq T_{ch}$ [9].

$T_{U(1)}$: the temperature at which the $U(1)$ axial symmetry is (effectively) restored, meaning that, for $T > T_{U(1)}$, there are no $U(1)$–breaking condensates. The Witten–Veneziano mechanism implies that $T_{U(1)} \geq T_\chi$, since, after all, the pure–YM topological susceptibility A is a $U(1)$–breaking condensate. Moreover, if $\langle \bar{q}q \rangle \neq 0$ also the $U(1)$ axial symmetry is broken, i.e., the chiral condensate is an order parameter also for the $U(1)$ axial symmetry. Therefore we must have: $T_{U(1)} \geq T_{ch}$. Present lattice results for the chiral susceptibilities indicate that $T_{U(1)} > T_{ch}$ [11–13].

Thus we need another quantity which could be an order parameter only for the $U(1)$ chiral symmetry [15–19]. The most simple quantity of this kind was found by 'tHooft in Ref. [2]. For a theory with L light quark flavours, it is a $2L$–fermion interaction that has the chiral transformation properties of:

$$\mathcal{L}_{eff} \sim \det_{st}(\bar{q}_{sR} q_{tL}) + \det_{st}(\bar{q}_{sL} q_{tR}), \qquad (4)$$

where $s, t = 1, \ldots, L$ are flavour indices, but the colour indices are arranged in a more general way (see Refs. [17–19]). It is easy to verify that $\mathcal{L}_{eff}$ is invariant under $SU(L) \otimes SU(L) \otimes U(1)_V$, while it is not invariant under $U(1)_A$. To obtain an order parameter for the $U(1)$ chiral symmetry, one can simply take the vacuum expectation value of $\mathcal{L}_{eff}$: $C_{U(1)} = \langle \mathcal{L}_{eff} \rangle$. The arbitrarity in the arrangement of the colour indices can be removed if we require that the new $U(1)$ chiral condensate is "independent" of the usual chiral condensate $\langle \bar{q}q \rangle$, as explained in Refs. [17–19]. In other words, the condensate $C_{U(1)}$ is chosen to be a *genuine* $2L$–fermion condensate, with a zero "disconnected part", the latter being the contribution proportional to $\langle \bar{q}q \rangle^L$, corresponding to retaining the vacuum intermediate state in all the channels and neglecting the contributions of all the other states. As a remark, we observe that the condensate $C_{U(1)}$ so defined turns out to be of order $\mathcal{O}(g^{2L-2} N_c^L) = \mathcal{O}(N_c)$ in the large–N_c expansion,

exactly as the chiral condensate $\langle \bar{q}q \rangle$.

The existence of a new $U(1)$ chiral order parameter has of course interesting physical consequences, which can be revealed by analysing some relevant QCD Ward Identities (WI's) (see Refs. [16,19]). In the case of the $SU(L) \otimes SU(L)$ chiral symmetry, one immediately derives the following WI:

$$\int d^4x \, \langle T \partial^\mu A^a_\mu(x) i\bar{q}\gamma_5 T^b q(0) \rangle = i\delta_{ab} \frac{1}{L} \langle \bar{q}q \rangle, \quad (5)$$

where $A^a_\mu = \bar{q}\gamma_\mu\gamma_5 T^a q$ are the $SU(L)$ axial currents. If $\langle \bar{q}q \rangle \neq 0$ (in the chiral limit $\sup(m_i) \to 0$), the anomaly–free WI (5) implies the existence of $L^2 - 1$ non–singlet Goldstone bosons, interpolated by the hermitian fields $O_b = i\bar{q}\gamma_5 T^b q$. Similarly, in the case of the $U(1)$ axial symmetry, one finds that:

$$\int d^4x \, \langle T \partial^\mu J_{5,\mu}(x) i\bar{q}\gamma_5 q(0) \rangle = 2i \langle \bar{q}q \rangle, \quad (6)$$

where $J_{5,\mu} = \bar{q}\gamma_\mu\gamma_5 q$ is the $U(1)$ axial current. But this is not the whole story! One also derives the following WI:

$$\int d^4x \, \langle T \partial^\mu J_{5,\mu}(x) O_P(0) \rangle = 2Li \langle \mathcal{L}_{eff}(0) \rangle, \quad (7)$$

where $\mathcal{L}_{eff}$ is the $2L$–fermion operator defined by Eq. (4), while the hermitian field O_P is defined as: $O_P \sim i[\det(\bar{q}_{sR}q_{tL}) - \det(\bar{q}_{sL}q_{tR})]$. If the $U(1)$–breaking condensate survives across the chiral transition at T_{ch}, i.e., $C_{U(1)} = \langle \mathcal{L}_{eff}(0) \rangle \neq 0$ for $T > T_{ch}$ (while $\langle \bar{q}q \rangle = 0$ for $T > T_{ch}$), then this WI implies the existence of a (*would–be*) Goldstone boson (in the large–N_c limit) coming from this breaking and interpolated by the hermitian field O_P. Therefore, the $U(1)_A$ (*would–be*) Goldstone boson (i.e., the η') is an "exotic" $2L$–fermion state for $T > T_{ch}$.

3. The new chiral effective Lagrangian

It is well known that the low–energy dynamics of the pseudoscalar mesons, including the effects due to the anomaly and the $q\bar{q}$ chiral condensate, can be described, in the large–N_c limit, and expanding to the first order in the light quark masses, by an effective Lagrangian [20–24] written in terms of the mesonic field $U_{ij} \sim \bar{q}_{jR}q_{iL}$ (up to a multiplicative constant) and the topological charge density Q. We make the assumption that there is a $U(1)$–breaking condensate which stays different from zero across T_{ch}, up to $T_{U(1)} > T_{ch}$: the form of this condensate has been discussed in the previous section. We must now define a field variable X, associated with this new condensate, to be inserted in the chiral Lagrangian. The operators $i\bar{q}\gamma_5 q$ and $\bar{q}q$ entering in the WI (6) are essentially equal to (up to a multiplicative constant) $i(\mathrm{Tr}U - \mathrm{Tr}U^\dagger)$ and $\mathrm{Tr}U + \mathrm{Tr}U^\dagger$ respectively. Similarly, the form of the new field X, in terms of the fundamental quark fields, can be derived from the WI (7), identifying the operators O_P and $\mathcal{L}_{eff}$ with (up to a multiplicative constant) $i(X - X^\dagger)$ and $X + X^\dagger$ respectively: this gives $X \sim \det(\bar{q}_{sR}q_{tL})$ (up to a multiplicative constant). It was shown in Refs. [15–17,19] that the most simple effective Lagrangian, constructed with the fields U, X and Q, is:

$$\begin{aligned}
&\mathcal{L}(U, U^\dagger, X, X^\dagger, Q) \\
&\quad = \frac{1}{2}\mathrm{Tr}(\partial_\mu U \partial^\mu U^\dagger) + \frac{1}{2}\partial_\mu X \partial^\mu X^\dagger \\
&\quad - V(U, U^\dagger, X, X^\dagger) + \frac{1}{2} iQ\omega_1 \mathrm{Tr}(\ln U - \ln U^\dagger) \\
&\quad + \frac{1}{2} iQ(1-\omega_1)(\ln X - \ln X^\dagger) + \frac{1}{2A}Q^2, \quad (8)
\end{aligned}$$

where the potential term $V(U, U^\dagger, X, X^\dagger)$ has the form:

$$\begin{aligned}
&V(U, U^\dagger, X, X^\dagger) \\
&\quad = \frac{\lambda_\pi^2}{4}\mathrm{Tr}[(U^\dagger U - \rho_\pi \mathbf{I})^2] + \frac{\lambda_X^2}{4}(X^\dagger X - \rho_X)^2 \\
&\quad - \frac{B_m}{2\sqrt{2}}\mathrm{Tr}(MU + M^\dagger U^\dagger) \\
&\quad - \frac{c_1}{2\sqrt{2}}[\det(U)X^\dagger + \det(U^\dagger)X]. \quad (9)
\end{aligned}$$

$M = \mathrm{diag}(m_1, \ldots, m_L)$ is the quark mass matrix. All the parameters appearing in the Lagrangian must be considered as functions of the physical temperature T. In particular, the parameters ρ_π and ρ_X determine the expectation values $\langle U \rangle$ and $\langle X \rangle$ and so they are responsible for the behaviour of the theory respectively across the $SU(L) \otimes SU(L)$ and the $U(1)$ chiral phase

transitions, as follows:

$$\rho_\pi|_{T<T_{ch}} \equiv \frac{1}{2}F_\pi^2 > 0, \quad \rho_\pi|_{T>T_{ch}} < 0;$$
$$\rho_X|_{T<T_{U(1)}} \equiv \frac{1}{2}F_X^2 > 0, \quad \rho_X|_{T>T_{U(1)}} < 0. \qquad (10)$$

The parameter F_π is the well–known pion decay constant, while the parameter F_X is related to the new $U(1)$ axial condensate and will be the object of our analysis. According to what we have said in the Introduction and in Section 2, we also assume that the topological susceptibility $A(T)$ of the pure–YM theory drops to zero at a temperature $T_\chi \geq T_{ch}$ (but $T_\chi \leq T_{U(1)}$).

One can study the mass spectrum of the theory for $T < T_{ch}$ and $T_{ch} < T < T_{U(1)}$. First of all, let us see what happens for $T < T_{ch}$, where both the $q\bar{q}$ chiral condensate and the $U(1)$ chiral condensate are present. Integrating out the field variable Q and taking only the quadratic part of the Lagrangian, one finds that, in the chiral limit $\sup(m_i) \to 0$, there are $L^2 - 1$ zero–mass states, which represent the L^2-1 Goldstone bosons coming from the breaking of the $SU(L)\otimes SU(L)$ chiral symmetry down to $SU(L)_V$. Then there are two singlet eigenstates with non–zero masses:

$$\eta' = \frac{1}{\sqrt{F_\pi^2 + LF_X^2}}(\sqrt{L}F_X S_X + F_\pi S_\pi),$$
$$\eta_X = \frac{1}{\sqrt{F_\pi^2 + LF_X^2}}(-F_\pi S_X + \sqrt{L}F_X S_\pi), \qquad (11)$$

where S_π is the usual "quark–antiquark" $SU(L)$–singlet meson field associated with U, while S_X is the "exotic" $2L$–fermion meson field associated with X [15,17,19]:

$$U = \frac{F_\pi}{\sqrt{2}} \exp\left[\frac{i\sqrt{2}}{F_\pi}\left(\sum_{a=1}^{L^2-1}\pi_a\tau_a + \frac{S_\pi}{\sqrt{L}}\mathbf{I}\right)\right],$$
$$X = \frac{F_X}{\sqrt{2}} \exp\left(\frac{i\sqrt{2}}{F_X}S_X\right). \qquad (12)$$

The matrices τ_a ($a = 1, \dots, L^2-1$) are the generators of the algebra of $SU(L)$ in the fundamental representation, with normalization: $\mathrm{Tr}(\tau_a\tau_b) = \delta_{ab}$. The π_a are the self–hermitian fields describing the $L^2 - 1$ massless pions.

The field η' has a "light" mass, in the sense of the $N_c \to \infty$ limit, being

$$m_{\eta'}^2 = \frac{2LA}{F_\pi^2 + LF_X^2} = \mathcal{O}(\frac{1}{N_c}). \qquad (13)$$

On the contrary, the field η_X has a sort of "heavy hadronic" mass of order $\mathcal{O}(N_c^0)$ in the large–N_c limit. Both the η' and the η_X have the same quantum numbers (spin, parity and so on), but they have a different quark content: one is mostly $S_\pi \sim i(\bar{q}_L q_R - \bar{q}_R q_L)$, while the other is mostly $S_X \sim i[\det(\bar{q}_{sL}q_{tR}) - \det(\bar{q}_{sR}q_{tL})]$. What happens when approaching the chiral transition temperature T_{ch}? We know that $F_\pi(T) \to 0$ when $T \to T_{ch}$. From Eq. (13) we see that $m_{\eta'}^2(T_{ch}) = \frac{2A}{F_X^2}$ and, from the first Eq. (11), $\eta'(T_{ch}) = S_X$. We have continuity in the mass spectrum of the theory through the chiral phase transition at $T = T_{ch}$. In fact, if we study the mass spectrum of the theory in the region of temperatures $T_{ch} < T < T_{U(1)}$ (where the $SU(L)\otimes SU(L)$ chiral symmetry is restored, while the $U(1)$ chiral condensate is still present), one finds that there is a singlet meson field S_X (associated with the field X in the chiral Lagrangian) with a squared mass given by (in the chiral limit): $m_{S_X}^2 = \frac{2A}{F_X^2}$. This is nothing but the *would–be* Goldstone particle coming from the breaking of the $U(1)$ chiral symmetry, i.e., the η', which, for $T > T_{ch}$, is a sort of "exotic" matter field of the form $S_X \sim i[\det(\bar{q}_{sL}q_{tR}) - \det(\bar{q}_{sR}q_{tL})]$. Its existence could be proved perhaps in the near future by heavy–ion experiments.

4. A relation between χ' and the new $U(1)$ chiral condensate

In this section and in the following one we want to describe some methods which provide us with some information about the parameter F_X [14]. This quantity is a $U(1)$–breaking parameter: indeed, from Eq. (10), $\rho_X = \frac{1}{2}F_X^2 > 0$ for $T < T_{U(1)}$, and therefore, from Eq. (9), $\langle X \rangle = F_X/\sqrt{2} \neq 0$. Remembering that $X \sim \det(\bar{q}_{sR}q_{tL})$, up to a multiplicative constant, we find that F_X is proportional to the new $2L$–fermion condensate $C_{U(1)} = \langle \mathcal{L}_{eff} \rangle$ introduced above.

In the same way, the pion decay constant F_π, which controls the breaking of the $SU(L)\otimes SU(L)$ symmetry, is related to the $q\bar{q}$ chiral condensate by a simple and well–known proportionality relation (see Refs. [15,19] and references therein): $\langle \bar{q}_i q_i \rangle_{T<T_{ch}} \simeq -\frac{1}{2} B_m F_\pi$. Considering, for simplicity, the case of L light quarks with the same mass m, one immediately derives from this equation the so–called *Gell-Mann–Oakes–Renner relation*: $m_{NS}^2 F_\pi^2 \simeq -\frac{2m}{L} \langle \bar{q} q \rangle_{T<T_{ch}}$, where, as usual, $\langle \bar{q} q \rangle \equiv \sum_{i=1}^L \langle \bar{q}_i q_i \rangle$, and, moreover, $m_{NS}^2 = m B_m / F_\pi$ is the squared mass of the non–singlet pseudoscalar mesons.

It is not possible to find, in a simple way, the analogous relation between F_X and the new condensate $C_{U(1)} = \langle \mathcal{L}_{eff} \rangle$.

Alternatively, the quantity F_X can be extracted from the two–point Green function of the topological charge–density operator $Q(x)$ in the *full* theory with L light quarks:

$$\chi(k) \equiv -i \int d^4x \, e^{ikx} \langle TQ(x)Q(0) \rangle. \qquad (14)$$

The calculation of $\chi(k)$ can be performed explicitly, using our effective Lagrangian. The most interesting result is found when considering the so–called "slope" of the topological susceptibility, defined as:

$$\chi' \equiv \frac{1}{8} \frac{\partial}{\partial k_\mu} \frac{\partial}{\partial k^\mu} \chi(k) \bigg|_{k=0} = \frac{d}{dk^2} \chi(k) \bigg|_{k=0} = \frac{i}{8} \int d^4x \, x^2 \langle TQ(x)Q(0) \rangle, \qquad (15)$$

which, in the chiral limit of L *massless* quarks, comes out to be, for $T < T_{ch}$ [14]:

$$\chi'_{ch} = -\frac{1}{2L}(F_\pi^2 + L F_X^2) \equiv -\frac{1}{2L} F_{\eta'}^2, \qquad (16)$$

where $F_{\eta'} \equiv \sqrt{F_\pi^2 + L F_X^2}$ is the decay constant of the η' (at the leading order in the $1/N_c$ expansion), modified by the presence of the new $U(1)$ chiral order parameter [17,19]. In fact, remembering how the fields U and X transform under a $U(1)$ chiral transformation, one can determine the $U(1)$ axial current, starting from our effective Lagrangian [17,19]:

$$J_{5,\mu} = i \left[\mathrm{Tr}(U^\dagger \partial_\mu U - U \partial_\mu U^\dagger) + L(X^\dagger \partial_\mu X - X \partial_\mu X^\dagger) \right] = -\sqrt{2L} F_{\eta'} \partial_\mu \eta', \qquad (17)$$

where the field η' is defined by the first Eq. (11) and the relative coupling between $J_{5,\mu}$ and η', i.e., the $SU(L)$–singlet (η') decay constant defined as $\langle 0 | J_{5,\mu}(0) | \eta'(p) \rangle = i\sqrt{2L}\, p_\mu \, F_{\eta'}$, comes out to be:

$$F_{\eta'} = \sqrt{F_\pi^2 + L F_X^2}. \qquad (18)$$

Summarizing, we have found that the value of χ', in the chiral limit $\sup(m_i) \to 0$, is shifted from the "original" value $-\frac{1}{2L} F_\pi^2$ (derived in the absence of an extra $U(1)$ chiral condensate: see Refs. [25,26]) to the value $-\frac{1}{2L} F_{\eta'}^2 = -\frac{1}{2L}(F_\pi^2 + L F_X^2)$, which also depends on the quantity F_X, proportional to the extra $U(1)$ chiral condensate.

All the above refers to the theory at $T < T_{ch}$. When approaching the chiral transition at $T = T_{ch}$, one expects that F_π vanishes, while F_X remains different from zero and the quantity χ'_{ch} tends to the value:

$$\chi'_{ch} \underset{T \to T_{ch}}{\longrightarrow} -\frac{1}{2} F_X^2. \qquad (19)$$

The quantity $\chi(k)$ can also be evaluated in the region of temperatures $T_{ch} < T < T_{U(1)}$, proceeding as for the case $T < T_{ch}$, obtaining the result (already derived in Ref. [15]):

$$\chi(k) = A \frac{k^2}{k^2 - \frac{2A}{F_X^2}}, \qquad (20)$$

in the chiral limit $\sup(m_i) \to 0$.

Therefore, in the region of temperatures $T_{ch} < T < T_{U(1)}$, χ'_{ch} is given by [14]:

$$\chi'_{ch} = \frac{d}{dk^2} \chi(k) \bigg|_{k=0} = -\frac{1}{2} F_X^2, \qquad (21)$$

consistently with the results (16) and (19) found above: i.e., χ'_{ch} varies with continuity across T_{ch}. This means that χ'_{ch} acts as a sort of order parameter for the $U(1)$ axial symmetry above T_{ch}: if χ'_{ch} is different from zero above T_{ch}, this means that the $U(1)$–breaking parameter F_X is different from zero.

5. Radiative decays of the pseudoscalar mesons

Further information on the quantity F_X (i.e., on the new $U(1)$ chiral condensate, to which it is

related) can be derived from the study of the radiative decays of the "light" pseudoscalar mesons in two photons, $\pi^0, \eta, \eta', \eta_X \to \gamma\gamma$, in the realistic case of $L = 3$ light quarks (with non–zero masses m_u, m_d and m_s) and in the simple case of zero temperature ($T = 0$) [14].
To this purpose, we have to introduce the electromagnetic interaction in our effective model. Under *local* $U(1)$ electromagnetic transformations:

$$q \to q' = e^{i\theta e \mathbf{Q}} q, \quad A_\mu \to A'_\mu = A_\mu - \partial_\mu \theta, \qquad (22)$$

the fields U and X transform as follows:

$$U \to U' = e^{i\theta e \mathbf{Q}} U e^{-i\theta e \mathbf{Q}}, \quad X \to X' = X. \qquad (23)$$

Therefore, we have to replace the derivative of the fields $\partial_\mu U$ and $\partial_\mu X$ with the corresponding *covariant* derivatives:

$$D_\mu U = \partial_\mu U + ie A_\mu [\mathbf{Q}, U], \quad D_\mu X = \partial_\mu X. \qquad (24)$$

Here $\mathbf{Q}$ is the quark charge matrix (in units of e, the absolute value of the electron charge):

$$\mathbf{Q} = \begin{pmatrix} \frac{2}{3} & & \\ & -\frac{1}{3} & \\ & & -\frac{1}{3} \end{pmatrix}. \qquad (25)$$

In addition, we have to reproduce the effects of the electromagnetic anomaly, whose contribution to the four–divergence of the $U(1)$ axial current ($J_{5,\mu} = \bar{q}\gamma_\mu\gamma_5 q$) and of the $SU(3)$ axial currents ($A^a_\mu = \bar{q}\gamma_\mu\gamma_5 \frac{\tau_a}{\sqrt{2}} q$) is given by:

$$(\partial^\mu J_{5,\mu})^{e.m.}_{anomaly} = 2\mathrm{Tr}(\mathbf{Q}^2) G,$$
$$(\partial^\mu A^a_\mu)^{e.m.}_{anomaly} = 2\mathrm{Tr}\left(\mathbf{Q}^2 \frac{\tau_a}{\sqrt{2}}\right) G, \qquad (26)$$

where $G \equiv \frac{e^2 N_c}{32\pi^2} \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$ ($F_{\mu\nu}$ being the electromagnetic field–strength tensor), thus breaking the corresponding chiral symmetries. We observe that $\mathrm{Tr}(\mathbf{Q}^2 \tau_a) \neq 0$ only for $a = 3$ or $a = 8$.
We must look for an interaction term $\mathcal{L}_I$ (constructed with the chiral Lagrangian fields and the electromagnetic operator G) which, under a $U(1)$ axial transformation $q \to q' = e^{-i\alpha\gamma_5} q$, transforms as:

$$U(1)_A : \quad \mathcal{L}_I \to \mathcal{L}_I + 2\alpha \mathrm{Tr}(\mathbf{Q}^2) G, \qquad (27)$$

while, under $SU(3)$ axial transformations of the type $q \to q' = e^{-i\beta\gamma_5\tau_a/\sqrt{2}} q$ (with $a = 3, 8$), transforms as:

$$SU(3)_A : \quad \mathcal{L}_I \to \mathcal{L}_I + 2\beta \mathrm{Tr}\left(\mathbf{Q}^2 \frac{\tau_a}{\sqrt{2}}\right) G. \qquad (28)$$

By virtue of the transformation properties of the fields U and X under a $U(L) \otimes U(L)$ chiral transformation [15,19], one can see that the most simple term describing the electromagnetic anomaly interaction term is the following one:

$$\mathcal{L}_I = \frac{1}{2} i G \mathrm{Tr}[\mathbf{Q}^2 (\ln U - \ln U^\dagger)], \qquad (29)$$

which is exactly the one originally proposed in Ref. [27]. Therefore, we have to consider the following effective chiral Lagrangian, which includes the electromagnetic interaction terms described above [14]:

$$\begin{aligned} &\mathcal{L}(U, U^\dagger, X, X^\dagger, Q, A^\mu) \\ &\quad = \frac{1}{2}\mathrm{Tr}(D_\mu U D^\mu U^\dagger) + \frac{1}{2}\partial_\mu X \partial^\mu X^\dagger \\ &\quad -V(U, U^\dagger, X, X^\dagger) + \frac{1}{2} i Q\, \omega_1 \mathrm{Tr}(\ln U - \ln U^\dagger) \\ &\quad +\frac{1}{2} i Q (1 - \omega_1)(\ln X - \ln X^\dagger) + \frac{1}{2A} Q^2 \\ &\quad +\frac{1}{2} i G \mathrm{Tr}[\mathbf{Q}^2 (\ln U - \ln U^\dagger)] - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \end{aligned} \qquad (30)$$

The decay amplitude of the generic process "*meson* $\to \gamma\gamma$" is entirely due to the electromagnetic anomaly interaction term, which can be written more explicitly as follows, in terms of the meson fields:

$$\mathcal{L}_I = -G \frac{1}{3F_\pi}\left(\pi_3 + \frac{1}{\sqrt{3}}\pi_8 + \frac{2\sqrt{2}}{\sqrt{3}} S_\pi\right). \qquad (31)$$

The fields π_3, π_8, S_π, S_X mix together. However, neglecting the experimentally small mass difference between the quarks *up* and *down* (i.e., neglecting the experimentally small violations of the $SU(2)$ isotopic spin), also π_3 becomes diagonal and can be identified with the physical state π^0. The fields (π_8, S_π, S_X) can be written in terms of the eigenstates (η, η', η_X) as follows:

$$\begin{pmatrix} \pi_8 \\ S_\pi \\ S_X \end{pmatrix} = \mathbf{C} \begin{pmatrix} \eta \\ \eta' \\ \eta_X \end{pmatrix}, \qquad (32)$$

where $\mathbf{C}$ is the following 3×3 orthogonal matrix:

$$\mathbf{C} = \begin{pmatrix} \cos\tilde{\varphi} & -\sin\tilde{\varphi} & 0 \\ \sin\tilde{\varphi}\,\frac{F_\pi}{F_{\eta'}} & \cos\tilde{\varphi}\,\frac{F_\pi}{F_{\eta'}} & \frac{\sqrt{3}F_X}{F_{\eta'}} \\ \sin\tilde{\varphi}\,\frac{\sqrt{3}F_X}{F_{\eta'}} & \cos\tilde{\varphi}\,\frac{\sqrt{3}F_X}{F_{\eta'}} & -\frac{F_\pi}{F_{\eta'}} \end{pmatrix}. \quad (33)$$

Here $F_{\eta'}$ is defined according to Eq. (18), i.e.,

$$F_{\eta'} \equiv \sqrt{F_\pi^2 + 3F_X^2}, \quad (34)$$

and $\tilde{\varphi}$ is a mixing angle, which can be related to the masses of the quarks m_u, m_d, m_s, and therefore to the masses of the octet mesons, by the following relation:

$$\tan\tilde{\varphi} = \frac{F_\pi F_{\eta'}}{6\sqrt{2}A}(m_\eta^2 - m_\pi^2), \quad (35)$$

where: $m_\pi^2 = 2B\tilde{m}$ and $m_\eta^2 = \frac{2}{3}B(\tilde{m} + 2m_s)$, with: $B \equiv \frac{B_m}{2F_\pi}$, $\tilde{m} \equiv \frac{m_u+m_d}{2}$. With simple standard calculations, the following decay rates (in the real case $N_c = 3$) are derived [14]:

$$\Gamma(\pi^0 \to \gamma\gamma) = \frac{\alpha^2 m_\pi^3}{64\pi^3 F_\pi^2},$$
$$\Gamma(\eta \to \gamma\gamma) = \frac{\alpha^2 m_\eta^3}{192\pi^3 F_\pi^2}\Big(\cos\tilde{\varphi} + 2\sqrt{2}\sin\tilde{\varphi}\,\frac{F_\pi}{F_{\eta'}}\Big)^2,$$
$$\Gamma(\eta' \to \gamma\gamma) = \frac{\alpha^2 m_{\eta'}^3}{192\pi^3 F_\pi^2}\Big(2\sqrt{2}\cos\tilde{\varphi}\,\frac{F_\pi}{F_{\eta'}} - \sin\tilde{\varphi}\Big)^2,$$
$$\Gamma(\eta_X \to \gamma\gamma) = \frac{\alpha^2 m_{\eta_X}^3}{8\pi^3 F_\pi^2}\Big(\frac{F_X}{F_{\eta'}}\Big)^2, \quad (36)$$

where $\alpha = e^2/4\pi \simeq 1/137$ is the fine–structure constant.

If we put $F_X = 0$ (i.e., if we neglect the new $U(1)$ chiral condensate), the expressions written above reduce to the corresponding ones derived in Ref. [27] using an effective Lagrangian which includes only the usual $q\bar{q}$ chiral condensate (so there is no field η_X!). The introduction of the new condensate (while leaving the $\pi^0 \to \gamma\gamma$ decay rate unaffected, as it must!) modifies the decay rates of η and η' (and, moreover, we also have to consider the particle η_X). In particular, it modifies the η' decay constant, already in the chiral limit $\sup(m_i) \to 0$, according to Eq. (34).

In conclusion, a study of the radiative decays $\eta \to \gamma\gamma$, $\eta' \to \gamma\gamma$ and a comparison with the experimental data can provide us with further information about the parameter F_X and the new exotic condensate. Using the experimental values for the various quantities which appear in the second and third Eq. (36), i.e.,

$$\begin{aligned} &F_\pi = 92.4(4)\ \text{MeV}, \\ &m_\eta = 547.30(12)\ \text{MeV}, \\ &m_{\eta'} = 957.78(14)\ \text{MeV}, \\ &\Gamma(\eta \to \gamma\gamma) = 0.46(4)\ \text{KeV}, \\ &\Gamma(\eta' \to \gamma\gamma) = 4.26(19)\ \text{KeV}, \end{aligned} \quad (37)$$

we can extract the following values for the quantity F_X and for the mixing angle $\tilde{\varphi}$ [14]:

$$F_X = 27(9)\ \text{MeV}, \quad \tilde{\varphi} = 16(3)^0. \quad (38)$$

Moreover, the values of F_X and $\tilde{\varphi}$ so found are perfectly consistent with the relation (35) for the mixing angle, if we use for the pure–YM topological susceptibility the value $A = (180 \pm 5\ \text{MeV})^4$, obtained from lattice simulations.

6. Conclusions

There are evidences from some lattice results that a new $U(1)$–breaking condensate survives across the chiral transition at T_{ch}, staying different from zero up to $T_{U(1)} > T_{ch}$. This scenario can be consistently reproduced using an effective Lagrangian model, which also includes the new $U(1)$ chiral condensate. This scenario could perhaps be verified in the near future by heavy–ion experiments, by analysing the pseudoscalar–meson spectrum in the singlet sector.

We have determined the effects due to the presence of the new $U(1)$ chiral order parameter on the slope of the topological susceptibility χ'_{ch}, in the *full* theory with L *massless* quarks. We have found that χ'_{ch} acts as an order parameter for the $U(1)$ axial symmetry above T_{ch} [14]. This prediction of our model could be tested in the near–future Monte Carlo simulations on the lattice: at present, lattice determinations of χ' only exist for the pure–gauge theory at $T = 0$, with gauge group $SU(2)$ [28] and $SU(3)$ [29] (but see

also Ref. [26] for a discussion about possible ambiguities in the definition of χ'_{ch} in a lattice regularized theory).

We have also investigated the effects of the new $U(1)$ chiral condensate on the radiative decays (at $T = 0$) of the pseudoscalar mesons η and η' in two photons. A first comparison of our results with the experimental data has been performed: the results are encouraging, pointing towards a certain evidence of a non–zero $U(1)$ axial condensate (i.e., $F_X \neq 0$) [14].

However, one should keep in mind that our results have been derived from a very simplified model, obtained doing a first–order expansion in $1/N_c$ and in the quark masses. We expect that such a model can furnish only qualitative or, at most, "semi–quantitative" predictions. When going beyond the leading order in $1/N_c$, it becomes necessary to take into account questions of renormalization–group behaviour of the various quantities and operators involved in our theoretical analysis. This issue has been widely discussed in the literature, both in relation to the analysis of χ'_{ch}, in the context of the proton–spin crisis problem [25], and also in relation to the study of the η, η' radiative decays [30]. Further studies are therefore necessary in order to continue this analysis from a more quantitative point of view.
Last, but not least, it would be also very interesting (for a comparison with future heavy–ion experiments) to extend our present analysis of the radiative decays to the non–zero–temperature case. We expect that some progress will be done along this line in the near future.

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